

IMPOSSIBLE INTERSECTIONS IN A WEIERSTRASS FAMILY OF ELLIPTIC CURVES

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ABSTRACT. Consider the Weierstrass family of elliptic curves $E_\lambda : y^2 = x^3 + \lambda$ parametrized by nonzero $\lambda \in \overline{\mathbb{Q}_2}$, and let $P_\lambda(x) = (x, \sqrt{x^3 + \lambda}) \in E_\lambda$. In this article, given $\alpha, \beta \in \overline{\mathbb{Q}_2}$ such that $\frac{\alpha}{\beta} \in \mathbb{Q}$, we provide an explicit description for the set of parameters λ such that $P_\lambda(\alpha)$ and $P_\lambda(\beta)$ are simultaneously torsion for E_λ . In particular we prove that the aforementioned set is empty unless $\frac{\alpha}{\beta} \in \{-2, -\frac{1}{2}\}$. Furthermore, we show that this set is empty even when $\frac{\alpha}{\beta} \notin \mathbb{Q}$ provided that α and β have distinct 2-adic absolute values and the ramification index $e(\mathbb{Q}_2(\frac{\alpha}{\beta}) \mid \mathbb{Q}_2)$ is coprime with 6. We also improve upon a recent result of Stoll concerning the Legendre family of elliptic curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$, which itself strengthened earlier work of Masser and Zannier by establishing that provided a, b have distinct reduction modulo 2, the set $\{\lambda \in \mathbb{C} \setminus \{0, 1\} : (a, \sqrt{a(a-1)(a-\lambda)}), (b, \sqrt{b(b-1)(b-\lambda)}) \in (E_\lambda)_{tors}\}$ is empty.

1. INTRODUCTION

Let $E_\lambda : y^2 = x(x-1)(x-\lambda)$ be the Legendre family of elliptic curves parametrized by $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and let $P_\lambda = (2, \sqrt{2(2-\lambda)}), Q_\lambda = (3, \sqrt{6(3-\lambda)}) \in E_\lambda$. Masser and Zannier [MZ08, MZ10] proved that the set of parameters λ such that both P_λ and Q_λ are torsion for E_λ is finite. Later, in [MZ12] they strengthened the result and showed that 2 and 3 are not special. More specifically, they proved that for any $P_\lambda, Q_\lambda \in E_\lambda$ with x coordinates in $\overline{\mathbb{C}(\lambda)}$ there are only finitely many λ such that both P_λ and Q_λ are torsion for E_λ unless there exist $n, m \in \mathbb{Z}$, not both 0, such that $[m]P_\lambda = [n]Q_\lambda$ for all $\lambda \in \mathbb{C}$. Moreover, they proved similar finiteness results for any fibred product of two elliptic curves [MZ14].

Recently, Stoll [Sto14] proved that in the case of the Legendre family of elliptic curves, given two sections with x -coordinates $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}$ that have different reductions ‘modulo 2’, the only possible parameters λ such that both are torsion for E_λ are α and β . Thus, Stoll improved upon results in [MZ08, MZ10]. The approach from [Sto14] involves a careful analysis of the 2-adic behavior of the n -th reduced division polynomial of E_λ . Furthermore, this approach provides a partial result towards the characterization of the set of parameters (μ, λ) for which three different points are torsion for $E_{\mu, \lambda} : y^2 = x^3 + \mu x + \lambda$, assuming λ is integral at 2. In [Sto14, Proposition 7] he describes the 2-adic behavior of the n -th reduced division polynomial of $E_{\mu, \lambda}$. However when $\mu = 0$ this result does not

Key words and phrases. attracting fixed point, elliptic curve, family of Lattès maps, impossible intersections, preperiodic point, torsion.

give precise information on the parameters λ for which a point with constant x -coordinate is torsion for $E_{0,\lambda} : y^2 = x^3 + \lambda$. It is primarily this situation that we aim to address here.

In this article we are mainly interested in a Weierstrass family of elliptic curves $E_\lambda : y^2 = x^3 + \lambda$, parametrized by $\lambda \in \mathbb{C}_2 \setminus \{0\}$, where \mathbb{C}_2 denotes the completion of $\overline{\mathbb{Q}_2}$ with respect to the 2-adic absolute value. Letting $T(\alpha)$ denote the set of all parameters $\lambda \in \mathbb{C}_2$ such that $(\alpha, \sqrt{\alpha^3 + \lambda})$ is torsion for E_λ , we establish the following theorem which is one of our main results.

Theorem 1.1. *If $\alpha, \beta \in \overline{\mathbb{Q}_2} \setminus \{0\}$ are such that $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$, then $T(\alpha) \cap T(\beta) = \emptyset$. Moreover, for all $a \in \overline{\mathbb{Q}_2} \setminus \{0\}$ we have $T(a) \cap T(-2a) = \{-a^3\}$.*

In order to derive Theorem 1.1, we study the 2-adic absolute values of the elements in $T(\alpha)$. Our methods are dynamical; we work with an associated family of Lattès maps on \mathbb{P}^1 , taking a quotient of the multiplication by-2 map on E_λ . With this approach, in Theorem 4.4 and Corollary 4.6, we present an alternative proof and minor strengthening of Stoll's result concerning the Legendre family of elliptic curves. Furthermore, the method applies to other families of rational maps on \mathbb{P}^1 , which we illustrate with a non-Lattès example, $f_\lambda(z) = \frac{z^d + \lambda}{pz}$, for integer $d \geq 2$ and prime $p \in \mathbb{Z}$.

Theorem 1.2. *Let $p \in \mathbb{Z}$ be a prime and consider the natural reduction map $\rho : \mathbb{P}^1(\mathbb{C}_p) \rightarrow \mathbb{P}^1(\overline{\mathbb{F}_p})$. For $d \in \mathbb{Z}_{\geq 2}$, let $f_\lambda(z) = \frac{z^d + \lambda}{pz}$. If $\alpha, \beta \in \mathbb{C}_p \setminus \{0\}$ are such that $\rho(\alpha) \neq \rho(\beta)$, then there is no parameter $\lambda \in \mathbb{C}_p$ for which α and β are both preperiodic for f_λ .*

To put our results in the appropriate context, we will highlight some key features of earlier work. Masser and Zannier's original approach in [MZ08, MZ10, MZ12] involved a key recent result by Pila and Zannier [PZ08] and relied strongly on the existence of the analytic uniformization map for an elliptic curve. They further pointed out a dynamical reformulation of the question based on the fact that a point with x coordinate a is torsion for E_λ if and only if a is a preperiodic point for the Lattès map induced by the multiplication by 2 map in E_λ (see 2.1 for the definition of a Lattès map). Using this reformulation and the equidistribution results of [BR06, C-L06, FRL06], DeMarco, Wang and Ye [DWY13] generalized the aforementioned result for points of small canonical height. Also, motivated by these results and replacing the family of Lattès maps by other families of rational maps, many results concerning the finiteness of the set of parameters such that both a and b are preperiodic for a 1-parameter family of rational maps have appeared in [BD11, GHT12, GHT15]. For an overview on the motivation for these results and an outline of the key ideas in the proofs, we refer the reader to [Z12].

As opposed to the approach in [MZ08, MZ10, MZ12] which uses a key recent result by Pila and Zannier [PZ08] and relies strongly on the existence of the analytic uniformization map for elliptic curves, and the approach from [BD11, DWY13, GHT12, GHT15], which uses the powerful equidistribution statements of Baker-Rumely [BR06], Yuan [Y08] and Yuan-Zhang [YZ10] for points of small height, our method, as outlined next, is much simpler.

The structure of this article is as follows. In Section 2, we consider a Weierstrass family given by $E_\lambda : y^2 = x^3 + \lambda$ where $\lambda \in \mathbb{C}_2 \setminus \{0\}$. We use the Lattès map $f_\lambda(z) = \frac{z^4 - 8\lambda z}{4(z^3 + \lambda)}$ induced by the duplication map on E_λ . More specifically, using the fact that

$$P_\lambda(\alpha) := (\alpha, \sqrt{\alpha^3 + \lambda}) \text{ is torsion for } E_\lambda \iff \alpha \text{ is a preperiodic point for } f_\lambda(z) = \frac{z^4 - 8\lambda z}{4(z^3 + \lambda)},$$

in Theorem 2.6, we provide a relation between the 2-adic absolute values of λ and α for $\lambda \in \mathbb{C}_2$ such that $P_\lambda(\alpha)$ is torsion for E_λ . This relation strongly depends on whether $0, \infty$ which are both 2-adically attracting fixed points for all f_λ , belong to the orbit of α under f_λ . Furthermore, it is a useful step towards finding pairs $(\alpha, \beta) \in \mathbb{C}_2^2$ such that $T(\alpha) \cap T(\beta) = \emptyset$, which is what we consider subsequently. Using Theorem 2.6, we establish results of this flavour in Corollary 2.17. As a special case, we get that if $\alpha, \beta \in \mathbb{Q}_2^{unr}$ have distinct 2-adic absolute values and $\frac{\alpha^3}{\beta^3} \notin \{-8, -\frac{1}{8}\}$, then $T(\alpha) \cap T(\beta) = \emptyset$. An important property of the family of elliptic curves and corresponding Lattès maps that we exploit in the proof of Theorem 2.6, is that they are isotrivial (see 2.4 and 2.2.).

In Section 3, building on the results obtained in Section 2, we proceed to establish Theorem 1.1. More precisely, we use Lemma 3.4 to reduce the question to proving the coprimality of certain families of polynomials, which can be done by elementary means.

In Section 4, to further demonstrate the efficacy of our approach, as well as the fact that our method does not rely on isotriviality in general, we apply our method to give a shorter proof of Stoll's result. As a by-product of our method, in Theorem 4.4 and Corollary 4.6, we obtain a slight strengthening of Stoll's original result, [Sto14, Corollary 4].¹

Finally, we conclude with a discussion on other families of maps which are not Lattès in Section 5, where we establish Theorem 1.2.

2. A WEIERSTRASS FAMILY: A TRICHOTOMY AND SOME IMPOSSIBLE INTERSECTIONS

Let \mathbb{C}_2 denote the completion of $\overline{\mathbb{Q}_2}$ with respect to the 2-adic absolute value, and let $E_\lambda : y^2 = x^3 + \lambda$ where $\lambda \in \mathbb{C}_2 \setminus \{0\}$.

Before proceeding to our results, we will give some definitions. First, we will define the notion of a Lattès map, which plays an important role for our purposes. For a survey on its various remarkable properties, we refer the reader to [M06] and [S07, 6].

Definition 2.1. [S07, Section 6.4] *A rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d \geq 2$ is called a Lattès map if there are an elliptic curve E , a morphism $\psi : E \rightarrow E$, and a finite separable morphism $\pi : E \rightarrow \mathbb{P}^1$ such that the following diagram is commutative.*

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1. \end{array}$$

¹While this article was under review, Stoll, in [Sto14v2, Proposition 7], obtained a stronger result that covers our Theorem 4.4 and Corollary 4.6 by incorporating elements of our approach in his proof.

In the following, we will use the Lattès maps induced by the multiplication by 2 on E_λ for $\lambda \neq 0$. More precisely, we will use the maps \mathbf{f}_λ defined by the commutative diagram below.

$$\begin{array}{ccc} E_\lambda & \xrightarrow{[2]} & E_\lambda \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\mathbf{f}_\lambda} & \mathbb{P}^1. \end{array}$$

Here $\pi : E_\lambda \rightarrow \mathbb{P}^1$ is the projection map onto the x -coordinate of the elliptic curve E_λ and $\mathbf{f}_\lambda : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is given as $\mathbf{f}_\lambda([X : Y]) = [X^4 - 8\lambda XY^3 : 4Y(X^3 + \lambda Y^3)]$. We will mainly work with the de-homogenized version of \mathbf{f}_λ defined as $f_\lambda(z) = \frac{z^4 - 8\lambda z}{4(z^3 + \lambda)}$, and in this setting we identify ∞ with the point $[1 : 0] \in \mathbb{P}^1$.

Note that the family of elliptic curves $\{E_\lambda\}_{\{\lambda \in \mathbb{C}_2 \setminus \{0\}\}}$ may also be viewed as a single elliptic curve $E : y^2 = x^3 + t$ defined over the function field $\mathbb{C}_2(t)$, and the family of Lattès maps $\{\mathbf{f}_\lambda\}_{\{\lambda \in \mathbb{C}_2 \setminus \{0\}\}}$ may also be viewed as a single rational map $f \in \mathbb{C}_2(t)(z)$. One important property of this elliptic curve E and this rational function f that will aid to the proof of Theorem 3.3, is that they are isotrivial.

Definition 2.2. *Let K be an algebraically closed field. An elliptic curve E defined over the function field $K(t)$ is called isotrivial if there exists a finite extension L of $K(t)$ and an elliptic curve \mathcal{E} defined over K such that E is L -isomorphic to \mathcal{E} .*

Example 2.3. *For the elliptic curve $E : y^2 = x^3 + t$ over $\mathbb{C}_2(t)$ and $\mathcal{E} : y^2 = x^3 + 1$ over \mathbb{C}_2 , we have a $\mathbb{C}_2(t^{1/2}, t^{1/3})$ -isomorphism as follows.*

$$\begin{aligned} \mathcal{E} &\rightarrow E \\ (x, y) &\mapsto (xt^{1/3}, yt^{1/2}). \end{aligned}$$

Definition 2.4. *Let K be an algebraically closed field. A rational function $\phi \in K(t)(z)$ is called isotrivial if there exists a finite extension L of $K(t)$ and a Möbius map $M \in \text{PGL}(2, L)$ such that $M^{-1} \circ \phi \circ M \in K(z)$.*

Example 2.5. *For the rational function $f(z) = \frac{z^4 - 8tz}{4(z^3 + t)} \in \mathbb{C}_2(t)(z)$, we have that if $M(z) = t^{1/3}z \in \text{PGL}(2, \mathbb{C}_2(t^{1/3}))$ then*

$$M^{-1} \circ f \circ M(z) = f_1(z) = \frac{z^4 - 8z}{4(z^3 + 1)} \in \mathbb{C}_2(z).$$

This equation reflects the fact that all E_λ , $\lambda \neq 0$ are isomorphic, see [S07, Theorem 6.46]. We denote the 2-adic absolute value defined on \mathbb{C}_2 by $|\cdot|$. With this notation

$$|2| = \frac{1}{2}.$$

Moreover, we note that throughout this article we assume that

$$0 \in \mathbb{N}.$$

Let $\alpha, \lambda \in \mathbb{C}_2$. We denote the orbit of α under the action of f_λ by

$$\mathcal{O}_{f_\lambda}(\alpha) = \{f_\lambda^n(\alpha) : n \in \mathbb{N}\}.$$

Here, and in general in this article we write g^n for the n -th compositional iterate of a function g . Moreover, for $g \in \mathbb{C}_2(z)$, we write

$$\begin{aligned} \text{PrePer}(g) &= \{z \in \mathbb{C}_2 : z \text{ is preperiodic for } g\} \\ &= \{z \in \mathbb{C}_2 : \mathcal{O}_g(z) \text{ is a finite set}\}. \end{aligned}$$

Recall that

$$\begin{aligned} T(\alpha) &= \{\lambda \in \mathbb{C}_2 \setminus \{0\} : (\alpha, \sqrt{\alpha^3 + \lambda}) \in (E_\lambda)_{tors}\} \\ &= \{\lambda \in \mathbb{C}_2 \setminus \{0\} : \alpha \text{ is preperiodic for } f_\lambda\}. \end{aligned}$$

Finally, we note that $0, \infty$ are persistently preperiodic points for the family of rational maps f_λ , $\lambda \in \mathbb{C}_2 \setminus \{0\}$. Thus, in what follows we assume that

$$\alpha \neq 0, \infty.$$

2.1. A trichotomy.

Theorem 2.6. *Let $\lambda \in T(\alpha)$. Then either $\lambda \in \{-\alpha^3, \frac{\alpha^3}{8}\}$ or*

$$|\lambda| \in \{4|\alpha|^3, 4^{1-(1/4)^m}|\alpha|^3, 2^{2+(1/4)^m}|\alpha|^3 : m \in \mathbb{N}_{\geq 1}\}.$$

Moreover, exactly one of the following is true.

- (1) $|\lambda| = 4|\alpha|^3 \iff 0, \infty \notin \mathcal{O}_{f_\lambda}(\alpha)$.
- (2) $|\lambda| = 4^{1-(1/4)^m}|\alpha|^3$ for some $m \in \mathbb{N}_{\geq 1}$, or $\lambda = -\alpha^3 \iff \infty \in \mathcal{O}_{f_\lambda}(\alpha)$.
- (3) $|\lambda| = 2^{2+(1/4)^m}|\alpha|^3$ for some $m \in \mathbb{N}_{\geq 1}$, or $\lambda = \frac{\alpha^3}{8} \iff 0 \in \mathcal{O}_{f_\lambda}(\alpha)$.

The isotriviality of $f(z) = \frac{z^4 - 8tz}{4(z^3 + t)} \in \mathbb{C}_2(t)(z)$ will play an important role in the proof of this theorem. We find it worthwhile to point out that if $L(z) = (4t)^{1/3}z \in \text{PGL}(2, \mathbb{C}_2(t^{1/3}))$ then

$$(1) \quad L^{-1} \circ f \circ L(z) = \frac{z^4 - 2z}{4z^3 + 1} = g(z) \in \mathbb{C}_2(z).$$

The map g here is the Lattès map corresponding to the multiplication by 2 on the elliptic curve $y^2 = x^3 + \frac{1}{4}$, and has the property that it exhibits 2-adic good reduction (see [S07, Section 2.5] for definition).

For the rest of this section, we write

$$g(z) = \frac{z^4 - 2z}{4z^3 + 1} \in \mathbb{C}_2(z) \text{ as in (1).}$$

Theorem 2.6 will be a consequence of the following proposition.

Proposition 2.7. *Let $w \in \text{Preper}(g) \setminus \{0, \infty\}$. Then, exactly one of the following holds.*

- (1) $|w| = 1 \iff 0, \infty \notin \mathcal{O}_g(w)$.

- (2) $|w| = |4^{-1/3}|^{\frac{1}{4^m}}$ for some $m \in \mathbb{N}_{\geq 1}$, or $w^3 = -\frac{1}{4} \iff \infty \in \mathcal{O}_g(w)$.
 (3) $|w| = |2^{1/3}|^{\frac{1}{4^m}}$ for some $m \in \mathbb{N}_{\geq 1}$, or $w^3 = 2 \iff 0 \in \mathcal{O}_g(w)$.

Assume for the moment that the aforementioned proposition holds, for the sake of establishing Theorem 2.6.

Proof of Theorem 2.6. The proof is a consequence of the isotriviality of $f \in \mathbb{C}_2(t)(z)$ and Proposition 2.7 as follows. In view of (1), we get that for all $n \in \mathbb{N}$ and $\lambda, \alpha \in \mathbb{C}_2$

$$f_\lambda^n(\alpha) = (4\lambda)^{1/3} g^n \left(\frac{\alpha}{(4\lambda)^{1/3}} \right).$$

This implies $\mathcal{O}_{f_\lambda}(\alpha) = (4\lambda)^{1/3} \mathcal{O}_g \left(\frac{\alpha}{(4\lambda)^{1/3}} \right)$, and thus

$$\lambda \in T(\alpha) \iff \frac{\alpha}{(4\lambda)^{1/3}} \in \text{PrePer}(g).$$

Hence, to find the elements of $T(\alpha)$ it suffices to find the preperiodic points of g , as in Proposition 2.7. \square

We now return to the proof of Proposition 2.7. For this purpose, we will need the following lemmas, which exploit the fact that 0 and ∞ are both 2-adically attracting fixed points of the map g with multipliers -2 and 4 respectively, as we can see in the following remark.

Remark 2.8. For $z \in D(0, 1) = \{z \in \mathbb{C}_2 : |z| < 1\}$, we may write

$$g(z) = -2z - \frac{9z}{4} \sum_{n \geq 1} (-4z^3)^n.$$

Moreover, for $\phi(z) = \frac{1}{g(1/z)} = \frac{z^4 + 4z}{1 - 2z^3} \in \mathbb{C}_2(z)$ and $z \in D(0, 1)$ we have

$$\phi(z) = 4z + \frac{9z}{2} \sum_{n \geq 1} (2z^3)^n.$$

Lemma 2.9. *If $w \in D(0, 1)$, then as $n \rightarrow \infty$ both $g^n(w) \rightarrow 0$ and $\phi^n(w) \rightarrow 0$. In particular,*

- *if $w \in \text{PrePer}(g)$, then $g^m(w) = 0$ for some $m \in \mathbb{N}$.*
- *if $w \in \text{PrePer}(\phi)$, then $\phi^k(w) = 0$ for some $k \in \mathbb{N}$.*

Proof. In view of Remark 2.8, we have that if $w \in D(0, 1)$, then $|g(w)| \leq \max \left\{ \frac{|w|}{2}, |w|^4 \right\}$ and $|\phi(w)| \leq \max \left\{ \frac{|w|}{4}, |w|^4 \right\}$. Thus, we infer that as $n \rightarrow \infty$ both $g^n(w) \rightarrow 0$ and $\phi^n(w) \rightarrow 0$. The rest of the statement now follows. \square

Remark 2.10. The above lemma follows from a more general fact about maps $f \in K(z)$ with good reduction and having an attracting fixed point, where K is a local field, see [RB13, Lemma 2.3]. It implies that if $a \in \overline{K}$ is an attracting fixed point of f , then all the preperiodic points of f that lie in the residue class of a must map to a .

Lemma 2.11. *Let $n \in \mathbb{N}$ and $w \in D(0, 1)$. Then the following hold.*

- *If $|g(w)| = |2^{1/3}|^{\frac{1}{4^n}}$, then $|w|^4 = |2^{1/3}|^{\frac{1}{4^n}}$.*
- *If $|\phi(w)| = |4^{1/3}|^{\frac{1}{4^n}}$, then $|w|^4 = |4^{1/3}|^{\frac{1}{4^n}}$.*

Furthermore, if $n \in \mathbb{N}$ is the smallest integer such that $g^{n+1}(w) = 0$, then $|w|^{4^n} = |2^{1/3}|$. Similarly, if $n \in \mathbb{N}$ is the smallest integer such that $\phi^{n+1}(w) = 0$, then $|w|^{4^n} = |4^{1/3}|$.

Proof. Let $n \in \mathbb{N}$ and $w \in D(0, 1)$. Using our hypothesis and the Taylor expansion in Remark 2.8, we have

$$|g(w)| = \left| -2w - \frac{9w}{4} \sum_{n \geq 1} (-4w^3)^n \right| = |2^{1/3}|^{\frac{1}{4^n}}.$$

If $|w^3| \leq |2|$, using the ultrametric inequality, we infer that

$$|2^{1/3}|^{\frac{1}{4^n}} \leq \max\{|2w|, |9w^4|\} \leq |2|^{4/3},$$

which contradicts the fact that $|2| < 1$. Therefore, we must have $|w^3| > |2|$. Another application of the ultrametric inequality now yields that

$$|g(w)| = |w|^4 = |2^{1/3}|^{\frac{1}{4^n}},$$

as claimed in the statement of the lemma. Now, if $n \in \mathbb{N}$ is the smallest integer such that $g^{n+1}(w) = 0$, then $(g^n(w))^3 = 2$, and hence $|g^n(w)| = |2^{1/3}|$. Inductively, we get $|w|^{4^n} = |2^{1/3}|$.

The case of ϕ is similar. In view of Remark 2.8 and our hypothesis we have

$$|\phi(w)| = \left| 4w + \frac{9w}{2} \sum_{n \geq 1} (2w^3)^n \right| = |4^{1/3}|^{\frac{1}{4^n}}.$$

If $|w^3| \leq |4|$, the ultrametric inequality yields $|4^{1/3}|^{\frac{1}{4^n}} \leq \max\{|4w|, |9w^4|\} \leq |4|^{4/3}$, contradicting the fact that $|4| < 1$. Thus, $|w^3| > |4|$ and $|\phi(w)| = |w|^4 = |4^{1/3}|^{\frac{1}{4^n}}$. Now, if $n \in \mathbb{N}$ is the smallest integer such that $\phi^{n+1}(w) = 0$, then $(\phi^n(w))^3 = -4$, and hence $|\phi^n(w)| = |4^{1/3}|$. Inductively, we get $|w|^{4^n} = |4^{1/3}|$. This finishes the proof of the lemma. \square

We can now piece together the previous lemmas to prove Proposition 2.7.

Proof of Proposition 2.7. Let $w \in \text{PrePer}(g)$. We consider the cases $|w| = 1$, $|w| < 1$ and $|w| > 1$ separately.

If $|w| = 1$, then the ultrametric inequality yields that $|g^n(w)| = 1$ for all $n \in \mathbb{N}$ and in particular $0, \infty \notin \mathcal{O}_g(w)$.

If $|w| > 1$, then $z = \frac{1}{w} \in \text{PrePer}(\phi) \cap D(0, 1)$ and hence Lemma 2.9 yields $\phi^{m+1}(z) = 0$ for some $m \in \mathbb{N}$. This, by using Lemma 2.11, implies $|z|^{4^m} = |4^{1/3}|$ and hence $|w| = |4^{-1/3}|^{\frac{1}{4^m}}$. If, in particular, we have $m = 0$, then we immediately get $w^3 = -\frac{1}{4}$.

Finally, assume $|w| < 1$. By Lemma 2.9 we have that $g^{m+1}(w) = 0$ for some $m \in \mathbb{N}$. Lemma 2.11 now yields $|w| = |2^{1/3}|^{\frac{1}{4^m}}$. In the case $m = 0$ we get $w^3 = 2$. The proposition is now established. \square

We conclude this section with some related remarks.

Remark 2.12. We find it worthwhile to mention that g has three other 2-adically attracting fixed points, namely -1 , $-\xi$, and $-\xi^2$, each with multiplier -2 , where ξ is a cube root of unity. These points give information about the preperiodic points of g of flavor similar to the cases of 0 and ∞ . More specifically, for $w \in \text{PrePer}(g)$, the following hold.

- If $w \in D(-1, 1)$, then there exists an $m \in \mathbb{N}$ such that $g^{m+1}(w) = -1$. For the smallest such m , we have the equality $|w + 1| = |2^{1/3}|^{\frac{1}{4^m}}$.
- If $w \in D(-\xi, 1)$, then there exists an $m \in \mathbb{N}$ such that $g^{m+1}(w) = -\xi$. For the smallest such m , we have the equality $|w + \xi| = |2^{1/3}|^{\frac{1}{4^m}}$.
- If $w \in D(-\xi^2, 1)$, then there exists an $m \in \mathbb{N}$ such that $g^{m+1}(w) = -\xi^2$. For the smallest such m , we have the equality $|w + \xi^2| = |2^{1/3}|^{\frac{1}{4^m}}$.

The proof follows along the same lines as Proposition 2.7. We will briefly sketch the case of -1 . For $z \in D(-1, 1)$, we have

$$|g(z) + 1| = |-2(z + 1) - 6(z + 1)^2 - 16(z + 1)^3 - 43(z + 1)^4 + R(z + 1)|,$$

where $|R(z + 1)| \leq |z + 1|^5$. This implies that if $w \in \text{PrePer}(g) \cap D(-1, 1)$, then there exists a smallest $m \in \mathbb{N}$ such that $g^{m+1}(w) = -1$, which in turn yields $|w + 1|^4 = |2^{1/3}|^{\frac{1}{4^m}}$. For the later, notice that if $|g(w) + 1| \geq |2^{1/3}|$ then the ultrametric inequality yields $|g(w) + 1| = |w + 1|^4$.

Remark 2.13. Notice that g has infinitely many periodic points. One way to see this is by recalling that g is the Lattès map corresponding to the duplication map on $E : y^2 = x^3 + \frac{1}{4}$ and hence its periodic points are the x -coordinates of the points in $\cup_{n \in \mathbb{N}_{\geq 1}} E[2^n - 1]$.

Moreover, if $a \in \mathbb{C}_2 \setminus \{0, -1, -\xi, -\xi^2, \infty\}$ is a periodic point of $g \in \mathbb{C}_2(z)$, then $|a| = |a + 1| = |a + \xi| = |a + \xi^2| = 1$. To see this, note that otherwise by Lemma 2.6 and Remark 2.12, the orbit of a under the action of g meets a fixed point of g , contradicting the periodicity of a .

Remark 2.14. The only \mathbb{Q}_2 -preperiodic points of $g \in \mathbb{C}_2(z)$ are the \mathbb{Q}_2 -fixed points of g , that is 0, ∞ , and -1 . To see this note that if $z \in \mathbb{Q}_2 \cap \text{PrePer}(g) \setminus \{0, \infty, -1\}$ then either $|z| < 1$ or $|z| > 1$ or $|z + 1| < 1$, in which case Lemma 2.11 and Remark 2.12 yield that there exists an $n \in \mathbb{N}$ such that $|z| = |2^{1/3}|^{\frac{1}{4^n}}$ or $|z| = |4^{-1/3}|^{\frac{1}{4^n}}$ or $|z + 1| = |2^{1/3}|^{\frac{1}{4^n}}$ respectively, contradicting the fact that $z \in \mathbb{Q}_2$.

Remark 2.15. Observe that all the absolute values for $\lambda \in T(\alpha)$ that appear in Theorem 2.6 do indeed occur, from which it immediately follows that $T(\alpha)$ is an infinite set. To see this, it suffices to prove that all absolute values that appear in Proposition 2.7 for preperiodic points of g do indeed occur. As we have seen, -1 is a fixed point of g of absolute value 1.

Let $n \in \mathbb{N}$. To find $w \in \text{PrePer}(g)$ such that $|w| = |2^{1/3}|^{\frac{1}{4^n}}$, in view of Lemma 2.11, it suffices to find $w \in \mathbb{C}_2$ such that $g^{m+1}(w) = 0$ and $g^m(w) \neq 0$. This can be achieved for $w \in \mathbb{C}_2$ satisfying $(g^m(w))^3 = 2$. Analogously, to find $w \in \text{PrePer}(g)$ such that $|w| = |4^{-1/3}|^{\frac{1}{4^n}}$, by Lemma 2.11, it suffices to find $z \in \mathbb{C}_2$ such that $\phi^{m+1}(z) = 0$ and $\phi^m(z) \neq 0$. This can be achieved for $z \in \mathbb{C}_2$ satisfying $(\phi^m(z))^3 = -4$.

2.2. Some applications: impossible intersections. Let $\alpha, \beta \in \mathbb{C}_2$. Assuming the existence of $\lambda \in T(\alpha) \cap T(\beta)$, Theorem 2.6 allows us to compute an explicit list for the possible values of $\frac{|\alpha|}{|\beta|}$.

Corollary 2.16. *Assume that $T(\alpha) \cap T(\beta) \neq \emptyset$ and let*

$$X = \left\{ 1, 2^{\frac{1}{4^r}}, 2^{\frac{1}{3 \cdot 4^r}}, 2^{\frac{1}{3}(\frac{1}{4^r} - \frac{1}{4^s})}, 2^{\frac{2}{3 \cdot 4^r}}, 2^{\frac{2}{3}(\frac{1}{4^r} - \frac{1}{4^s})}, 4^{\frac{1}{3 \cdot 4^r}} 2^{\frac{1}{3 \cdot 4^s}} : r, s \in \mathbb{N}, r \neq s \right\}.$$

Then we have that either $\frac{|\alpha|}{|\beta|} \in X$ or $\frac{|\beta|}{|\alpha|} \in X$. Moreover, $\frac{|\alpha|}{|\beta|} = \frac{1}{2}$ or $\frac{|\alpha|}{|\beta|} = 2$ if and only if $\frac{\alpha^3}{\beta^3} = -8$ or $\frac{\alpha^3}{\beta^3} = -\frac{1}{8}$ respectively.

Proof. The proof follows immediately from Theorem 2.6. \square

A consequence now is that $T(\alpha) \cap T(\beta) = \emptyset$, for all α, β that ‘disagree’ with our list in Corollary 2.16. More specifically, we get the following theorem.

Theorem 2.17. *If $\alpha, \beta \in \overline{\mathbb{Q}_2}$ satisfy $\gcd\left(6, e(\mathbb{Q}_2(\frac{\alpha}{\beta})|\mathbb{Q}_2)\right) = 1$, $\left|\frac{\alpha}{\beta}\right| \neq 1$ and $\frac{\alpha^3}{\beta^3} \notin \{-8, -\frac{1}{8}\}$, then we have that $T(\alpha) \cap T(\beta) = \emptyset$. Moreover, $T(a) \cap T(-2a) = \{-a^3\}$ for all $a \in \mathbb{C}_2 \setminus \{0\}$.*

Proof. The proof follows combining the fact that for any $c \in \overline{\mathbb{Q}_2}$ if $e := e(\mathbb{Q}_2(c)|\mathbb{Q}_2)$, then $|c| \in 2^{\frac{\mathbb{Z}}{e}}$ with Corollary 2.16. To see that $T(a) \cap T(-2a) = \{-a^3\}$, note that by Theorem 2.6 we get that if $\lambda \in T(a) \cap T(-2a)$ then $\lambda = -a^3$, in which case we have $f_{-a^3}(a) = \infty$ and $f_{-a^3}(-2a) = 0$. \square

3. A WEIERSTRASS FAMILY: MORE IMPOSSIBLE INTERSECTIONS

We use our notation as in Section 2. Recall that by $|\cdot|$, we mean the 2-adic absolute defined on \mathbb{C}_2 . Theorem 2.17 raises the question whether we could describe $T(\alpha) \cap T(\beta)$ for $\alpha, \beta \in \mathbb{C}_2$ with equal 2-adic absolute values. In this section, towards partially answering this question, we aim to prove Theorem 1.1, which asserts that if we restrict our attention to $\alpha, \beta \in \overline{\mathbb{Q}_2}$ satisfying $\frac{\alpha}{\beta} \in \mathbb{Q}$, then there are no parameters λ such that both α, β are preperiodic for f_λ , unless $\frac{\alpha}{\beta} \in \{-2, -\frac{1}{2}\}$.

Before we state the main theorem of this section, a couple of remarks are in order.

Remark 3.1. The isotriviality of $f(z) \in \mathbb{C}_2(t)(z)$ implies that for all $\alpha, \lambda, z \in \mathbb{C}_2$, we have $f_{\lambda\alpha^3}(\alpha z) = \alpha f_\lambda(z)$. It easily follows that $\mathcal{O}_{f_{\lambda\alpha^3}}(\alpha) = \alpha \mathcal{O}_{f_\lambda}(1)$ and $T(\alpha) = \alpha^3 T(1)$.

Remark 3.2. From Remark 3.1, we get

$$\lambda \in T(\alpha) \cap T(\beta) \Leftrightarrow \frac{\lambda}{\beta^3} \in T(1) \cap T\left(\frac{\alpha}{\beta}\right).$$

In particular, $\#(T(\alpha) \cap T(\beta)) = \#(T(1) \cap T\left(\frac{\alpha}{\beta}\right))$.

For the following we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_2$. For the reader's convenience, we now restate Theorem 1.1.

Theorem 3.3. *If $\alpha, \beta \in \overline{\mathbb{Q}}_2 \setminus \{0\}$ are such that $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$, then $T(\alpha) \cap T(\beta) = \emptyset$. Moreover, for all $a \in \overline{\mathbb{Q}}_2 \setminus \{0\}$ we have $T(a) \cap T(-2a) = \{-a^3\}$.*

As we have already seen in Theorem 2.17, when $\alpha, \beta \in \overline{\mathbb{Q}}_2$ with $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$ and $|\alpha| \neq |\beta|$, we have $T(\alpha) \cap T(\beta) = \emptyset$. Moreover when $a \in \overline{\mathbb{Q}}_2 \setminus \{0\}$, we have $T(a) \cap T(-2a) = \{-a^3\}$. Therefore, to prove Theorem 3.3, it suffices to show that $T(\alpha) \cap T(\beta) = \emptyset$ when $|\alpha| = |\beta|$. Our strategy will be to first show in Lemma 3.4 that if $\lambda \in T(\alpha) \cap T(\beta)$, then either $0 \in \mathcal{O}_{f_\lambda}(\alpha) \cap \mathcal{O}_{f_\lambda}(\beta)$ or $\infty \in \mathcal{O}_{f_\lambda}(\alpha) \cap \mathcal{O}_{f_\lambda}(\beta)$. Then, after proving the coprimality of certain polynomials in Lemmas 3.8 and 3.9, we will rule out these two cases as well.

Lemma 3.4. *Let $\alpha, \beta \in \mathbb{C}_2$ with $|\alpha| = |\beta|$ and $|\alpha - \beta| \leq \frac{|\alpha|}{2}$. Consider $\lambda \in T(\alpha) \cap T(\beta)$. Then either $0 \in \mathcal{O}_{f_\lambda}(\alpha) \cap \mathcal{O}_{f_\lambda}(\beta)$ or $\infty \in \mathcal{O}_{f_\lambda}(\alpha) \cap \mathcal{O}_{f_\lambda}(\beta)$.*

Proof. Let $\lambda \in T(\alpha) \cap T(\beta)$ and assume that $0, \infty \notin \mathcal{O}_{f_\lambda}(\alpha) \cap \mathcal{O}_{f_\lambda}(\beta)$. We want to show that this leads to a contradiction. In light of Theorem 2.6 and the fact that $|\alpha| = |\beta|$ we get that $0 \in \mathcal{O}_{f_\lambda}(\alpha)$ (respectively $\infty \in \mathcal{O}_{f_\lambda}(\alpha)$) if and only if $0 \in \mathcal{O}_{f_\lambda}(\beta)$ (respectively $\infty \in \mathcal{O}_{f_\lambda}(\beta)$). Our assumption thus implies that $0, \infty \notin \mathcal{O}_{f_\lambda}(\alpha) \cup \mathcal{O}_{f_\lambda}(\beta)$. For the rest of this proof we will denote $f_\lambda^n(\alpha)$ and $f_\lambda^n(\beta)$ by t_n and u_n respectively.

Since $\lambda \in T(\alpha) \cap T(\beta)$, we know that the sets

$$S = \{t_n : n \in \mathbb{N}\}, \text{ and } T = \{u_n : n \in \mathbb{N}\}$$

are finite. Therefore, the set $M = \{|t_n - u_n| : n \in \mathbb{N}\}$ is also finite.

We claim that $|t_n - u_n| = \frac{|\alpha - \beta|}{2^n}$ for all $n \in \mathbb{N}$. This will contradict the fact that M is finite, thus finishing our proof. We will now prove the claim using induction. For $n = 0$, we have $|t_0 - u_0| = |\alpha - \beta|$. For the inductive step, assume that the statement holds for some $n \geq 0$. Then

$$\begin{aligned} |t_{n+1} - u_{n+1}| &= 4 \left| \frac{t_n^4 - 8\lambda t_n}{t_n^3 + \lambda} - \frac{u_n^4 - 8\lambda u_n}{u_n^3 + \lambda} \right| \\ &= 4 \left| \frac{(t_n^4 - 8\lambda t_n)(u_n^3 + \lambda) - (u_n^4 - 8\lambda u_n)(t_n^3 + \lambda)}{(u_n^3 + \lambda)(t_n^3 + \lambda)} \right| \\ &= \frac{4}{|u_n^3 + \lambda| |t_n^3 + \lambda|} \left| t_n^3 u_n^3 (t_n - u_n) - \lambda(u_n^4 - t_n^4) + 8\lambda t_n u_n (t_n^2 - u_n^2) + 8\lambda^2 (u_n - t_n) \right|. \end{aligned}$$

By the induction hypothesis, we know that $|t_n - u_n| = \frac{|\alpha - \beta|}{2^n} \leq \frac{|\alpha|}{2^{n+1}}$. Moreover, since $\lambda \in T(\alpha) \cap T(\beta)$ and $0, \infty \notin \mathcal{O}_{f_\lambda}(\alpha) \cup \mathcal{O}_{f_\lambda}(\beta)$, we also have $\lambda \in T(t_n) \cap T(u_n)$ and $0, \infty \notin \mathcal{O}_{f_\lambda}(t_n) \cup \mathcal{O}_{f_\lambda}(u_n)$ for all $n \in \mathbb{N}$. Theorem 2.6 now yields that for all $n \in \mathbb{N}$, $|t_n| = |u_n| = \sqrt[3]{\frac{|\lambda|}{4}} = |\alpha|$. Therefore, we get that

$$\begin{aligned} |u_n^3 + \lambda| &= |t_n^3 + \lambda| = 4|\alpha|^3, \\ |t_n + u_n| &= |t_n - u_n + 2u_n| \leq \frac{|\alpha|}{2}, \text{ and} \\ |t_n^2 + u_n^2| &= |(t_n + u_n)^2 - 2t_n u_n| = \frac{|\alpha|^2}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} |t_n^3 u_n^3 (t_n - u_n)| &= |\alpha|^6 |t_n - u_n|, \\ |\lambda(u_n^4 - t_n^4)| &\leq |\alpha|^6 |t_n - u_n|, \\ |8\lambda t_n u_n (t_n^2 - u_n^2)| &\leq \frac{|\alpha|^6}{4} |t_n - u_n|, \\ |8\lambda^2 (t_n - u_n)| &= 2|\alpha|^6 |t_n - u_n|. \end{aligned}$$

Thus, using the induction hypothesis, we obtain

$$|t_{n+1} - u_{n+1}| = \frac{1}{2} |t_n - u_n| = \frac{|\alpha - \beta|}{2^{n+1}}.$$

This establishes our claim and concludes the proof. \square

Remark 3.5. We point out here that the condition $\frac{\alpha}{\beta} \in \mathbb{Q}$ in Theorem 3.3 has been made to ensure that when α, β have equal 2-adic absolute values, then $|\alpha - \beta| \leq \frac{|\alpha|}{2}$ holds. In this case we can apply Lemma 3.4.

To proceed with our proof, we need the following definition.

Definition 3.6. Given $a \in \mathbb{C} \setminus \{0\}$, we write $f_t^n(a) = \frac{A_n(a, t)}{B_n(a, t)}$, where $A_n(a, t), B_n(a, t) \in \mathbb{C}[t]$ are polynomials given recursively as

$$\begin{aligned} A_0(a, t) &= a, B_0(a, t) = 1. \\ A_{n+1}(a, t) &= A_n(a, t)^4 - 8tA_n(a, t)B_n(a, t)^3. \\ B_{n+1}(a, t) &= 4B_n(a, t)A_n(a, t)^3 + 4tB_n(a, t)^4. \end{aligned}$$

Lemma 3.7. Let $a \in \mathbb{C} \setminus \{0\}$. We have $\gcd(A_n(a, t), B_n(a, t)) = 1$ for all $n \in \mathbb{N}$. Moreover, $\deg_t(A_n(a, t)) = \deg_t(B_n(a, t)) = \frac{4^n - 1}{3}$ for all $n \in \mathbb{N}$.

Proof. The proof follows from an easy induction. Note that our hypothesis that $a \neq 0$ implies that $A_n(a, 0), B_n(a, 0) \neq 0$ and thus $t \nmid A_n(a, t), B_n(a, t)$ for all $n \in \mathbb{N}$. \square

Our aim in the sequel will be to prove that when $a, b \in \mathbb{Z}$, the polynomials $A_n(a, t)$ and $A_n(b, t)$ (respectively $B_n(a, t)$ and $B_n(b, t)$) are coprime. This will in turn imply that $0, \infty \notin \mathcal{O}_{f_\lambda}(a) \cap \mathcal{O}_{f_\lambda}(b)$, which combined with view of Remark 3.2 is what we need in order to conclude the proof of Theorem 3.3, when $\frac{\alpha}{\beta} = \frac{a}{b}$. To achieve this we will first establish a few key lemmas.

Lemma 3.8. *For all $a, b \in \mathbb{Z}$ for which there exists a prime $p \neq 2$ such that $p|a$ and $p \nmid b$, we have*

$$\gcd(A_n(a, t), A_n(b, t)) = \gcd(B_n(a, t), B_n(b, t)) = 1, \text{ for all } n \in \mathbb{N}.$$

Proof. We start by noting that given $g(t) \in \mathbb{Z}[t]$, we denote its reduction modulo p by $\overline{g}(t) \in \mathbb{F}_p[t]$, where p is the same prime as in the statement of the lemma. Moreover, we denote $a_n(a, t) = \frac{A_n(a, t)}{a}$. From the recursion in Definition 3.6 we see that $a_n(a, t) \in \mathbb{Z}[t]$. Our strategy is to first establish that

$$(2) \quad \gcd(\overline{a_n}(a, t), \overline{A_n}(b, t)) = \gcd(\overline{B_n}(a, t), \overline{B_n}(b, t)) = 1.$$

Since $a_n(a, t) \in \mathbb{Z}[t]$ and $p|a$, we get that $\overline{A_n}(a, t) = 0$ for all $n \in \mathbb{N}$. Therefore, the recursive relations in Definition 3.6 yield that

$$\overline{B_{n+1}}(a, t) = 4t\overline{B_n}(a, t)^4 \text{ and } \overline{B_1}(a, t) = 4t.$$

Thus, we obtain that $\overline{B_n}(a, t) = (4t)^{\frac{4^n-1}{3}}$. Additionally, we have the following equalities.

$$(3) \quad \overline{a_{n+1}}(a, t) = -2(4t)^{4^n} \overline{a_n}(a, t) \text{ and } \overline{a_1}(a, t) = -8t.$$

From (3), it follows that $\overline{a_n}(a, t) = (-2)^n (4t)^{\frac{4^n-1}{3}}$.

Observe now that, to derive (2), it suffices to prove that $\overline{A_n}(b, 0), \overline{B_n}(b, 0) \neq 0$. Notice that $A_n(b, 0) = b^{4^n}$ and $B_n(b, 0) = 4^n b^{4^n-1}$. Since $p \neq 2$ and $p \nmid b$, we have established the equality in (2). Now note that the degrees of the polynomials $a_n(a, t), A_n(b, t), B_n(a, t)$ and $B_n(b, t)$, as computed in Lemma 3.7, are equal to the degrees of their reductions modulo p . This, combined with (2) finishes the proof of the lemma. \square

We will prove another useful lemma of a similar flavor.

Lemma 3.9. *For all $n \in \mathbb{N}$, $\gcd(A_n(1, t), A_n(-1, t)) = \gcd(B_n(1, t), B_n(-1, t)) = 1$.*

Proof. We proceed in a similar fashion to the proof of the previous lemma, just that this time we reduce polynomials modulo 3. Throughout this proof we write $\overline{g}(t) \in \mathbb{F}_3[t]$ for the reduction modulo 3 of $g(t) \in \mathbb{Z}[t]$. We will show that

$$(4) \quad \gcd(\overline{A_n}(1, t), \overline{A_n}(-1, t)) = \gcd(\overline{B_n}(1, t), \overline{B_n}(-1, t)) = 1.$$

Inductively, we can prove that $\overline{A_n}(1, t) = \overline{B_n}(1, t)$ for all $n \in \mathbb{N}$. Therefore, using the recursion for $A_n(1, t)$, we get that $\overline{A_{n+1}}(1, t) = \overline{A_n}(1, t)^4(1+t)$. Thus,

$$\overline{A_n}(1, t) = \overline{B_n}(1, t) = (1+t)^{\frac{4^n-1}{3}}.$$

Moreover, inductively we can see that

$$\begin{aligned}\overline{A_n}(-1, t) &= (1 - t)^{\frac{4^n - 1}{3}}, \\ \overline{B_n}(-1, t) &= -(1 - t)^{\frac{4^n - 1}{3}}.\end{aligned}$$

This establishes (4). Since the degrees of the polynomials $A_n(1, t)$, $A_n(-1, t)$, $B_n(1, t)$ and $B_n(-1, t)$, computed in Lemma 3.7, are equal to the degrees of their reductions modulo 3, (4) yields the lemma. \square

We are now ready to prove Theorem 3.3. In the following, for a non-zero integer n and a prime p , we write

$$\exp_p(n) = \max\{e \in \mathbb{N} : p^e | n\}.$$

Proof of Theorem 3.3. Let $\alpha, \beta \in \overline{\mathbb{Q}_2}$ be such that $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$. We aim to prove that $T(\alpha) \cap T(\beta) = \emptyset$. In view of Theorem 2.17, we have that $T(\alpha) \cap T(\beta) = \emptyset$ if $|\alpha| \neq |\beta|$, unless $\frac{\alpha}{\beta} \notin \{-2, -\frac{1}{2}\}$. Assume now $|\alpha| = |\beta|$. We may write $\frac{\alpha}{\beta} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ are coprime and satisfy $|a| = |b| = 1$. By Remark 3.2 it suffices to prove that $T(a) \cap T(b) = \emptyset$. Assume to the contrary that $\lambda \in T(a) \cap T(b)$. Now Lemma 3.4 and Remark 3.5 together imply that either $0 \in \mathcal{O}_{f_\lambda}(a) \cap \mathcal{O}_{f_\lambda}(b)$ or $\infty \in \mathcal{O}_{f_\lambda}(a) \cap \mathcal{O}_{f_\lambda}(b)$.

Observe that $0 \in \mathcal{O}_{f_\lambda}(a) \cap \mathcal{O}_{f_\lambda}(b)$ if and only if $A_n(a, \lambda) = A_n(b, \lambda) = 0$ for some $n \in \mathbb{N}$, and $\infty \in \mathcal{O}_{f_\lambda}(a) \cap \mathcal{O}_{f_\lambda}(b)$ if and only if $B_n(a, \lambda) = B_n(b, \lambda) = 0$ for some $n \in \mathbb{N}$. Hence, to prove the theorem it suffices to show that

$$(5) \quad \gcd(A_n(a, t), A_n(b, t)) = \gcd(B_n(a, t), B_n(b, t)) = 1 \text{ for all } n \in \mathbb{N}.$$

To this end, we will consider two cases. If $a = -b = 1$, on invoking Lemma 3.9 we see that (5) follows. If on the other hand there exists a prime p such that $\exp_p(a) \neq \exp_p(b)$, since $|a| = |b| = 1$, we see that $p \neq 2$ and (5) again holds true by Lemma 3.8. \square

An immediate corollary of Theorem 3.3 is the following.

Corollary 3.10. *If $\alpha \in \mathbb{Q}$ and $\lambda \in T(\alpha)$, the orbit of α under the action of f_λ does not contain any rational number other than α except possibly 0 and ∞ .*

Proof. Let $\alpha \in \mathbb{Q}$ and $\lambda \in T(\alpha)$. Assume that for some $n \in \mathbb{N}$ we have $f_\lambda^n(\alpha) \in \mathbb{Q} \setminus \{0, \infty, \alpha\}$ to derive a contradiction. Observe that $\lambda \in T(\alpha) \cap T(f_\lambda^n(\alpha))$. This observation combined with Theorem 3.3 now yields that one of the following is true.

- $f_\lambda^n(\alpha) = -2\alpha$, in which case $\lambda = -\alpha^3$ and $f_\lambda(\alpha) = \infty$ contradicting the fact that $f_\lambda^n(\alpha) \neq \infty$.
 - $\alpha = -2f_\lambda^n(\alpha)$, in which case $\lambda = \frac{\alpha^3}{8}$ and $f_\lambda(\alpha) = 0$ contradicting the fact that $f_\lambda^n(\alpha) \neq 0$.
- In each case we derived a contradiction, yielding our claim. \square

We conclude this section with some observations and further questions.

Remark 3.11. Let $h : \overline{\mathbb{Q}} \rightarrow \mathbb{R}_{\geq 0}$ denote the absolute logarithmic Weil height, as in [S07, Section 3.1]. As seen from (1) we have $\lambda \in T(1) \Leftrightarrow \frac{1}{(4\lambda)^{1/3}} \in \text{PrePer}(g)$. Combining this with [S07, Theorem 3.12] we see that there is a constant $M > 0$ such that $h(\lambda) < M$ for all $\lambda \in T(1)$. Moreover, by Remark 3.1 we have $T(\alpha) \cap T(\beta) \neq \emptyset$ if and only if there exist $\lambda, \mu \in T(1)$ such that $\alpha^3 \lambda = \beta^3 \mu$. Therefore, $3h(\frac{\alpha}{\beta}) = h(\frac{\mu}{\lambda}) \leq 2M$. Hence, there is an absolute constant $C > 0$ such that $h(\frac{\alpha}{\beta}) < C$ for all $\alpha, \beta \in \overline{\mathbb{Q}}$ that satisfy $T(\alpha) \cap T(\beta) \neq \emptyset$. In particular there exist only finitely many $\frac{\alpha}{\beta}$ of bounded degree such that $T(\alpha) \cap T(\beta) \neq \emptyset$. In Theorem 3.3, we show that if $\frac{\alpha}{\beta} \in \mathbb{Q}$, then $T(\alpha) \cap T(\beta) \neq \emptyset$ if and only if $\frac{\alpha}{\beta} \in \{-2, -\frac{1}{2}\}$. This raises the following natural question: Fix $d \in \mathbb{Z}$, what is the best upper bound (depending on d) on the number of $\frac{\alpha}{\beta}$ of degree at most d such that $T(\alpha) \cap T(\beta) \neq \emptyset$?

Finally, it would be interesting to know whether there exist $\alpha, \beta \in \overline{\mathbb{Q}}$ such that $2 \leq \#(T(\alpha) \cap T(\beta)) < +\infty$. We note here that by Remark 3.1 if $\alpha^3 = \beta^3$, we have $T(\alpha) \cap T(\beta) = T(\alpha) = T(\beta)$, which by Remark 2.15 is an infinite set.

4. THE LEGENDRE FAMILY OF ELLIPTIC CURVES

For this section, let $E_\lambda : y^2 = x(x-1)(x-\lambda)$ be the Legendre family of elliptic curves parametrized by $\lambda \in \mathbb{C}_2 \setminus \{0, 1\}$. The Lattès map induced by the multiplication by 2 map on E_λ , is given as

$$f_\lambda(z) = \frac{(z^2 - \lambda)^2}{4z(z-1)(z-\lambda)}.$$

Let $\alpha \in \mathbb{C}_2$. We define $T(\alpha)$ as follows.

$$\begin{aligned} T(\alpha) &= \{\lambda \in \mathbb{C}_2 \setminus \{0, 1\} : (\alpha, \sqrt{\alpha(\alpha-1)(\alpha-\lambda)}) \in (E_\lambda)_{tors}\} \\ &= \{\lambda \in \mathbb{C}_2 \setminus \{0, 1\} : \alpha \text{ is preperiodic for } f_\lambda\}. \end{aligned}$$

First, we prove the following easy proposition.

Proposition 4.1. *For all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, we have $f_\lambda^n(\frac{1}{x}) = \frac{1}{\lambda} f_\lambda^n(x)$. In particular, $\lambda \in T(x)$ if and only if $\frac{1}{\lambda} \in T(\frac{1}{x})$.*

Proof. For $n = 1$, we have $f_\lambda(\frac{1}{x}) = \frac{1}{\lambda} f_\lambda(x)$. Using the easily verifiable fact $f_\lambda(\frac{z}{\lambda}) = \frac{1}{\lambda} f_\lambda(z)$, we get the result inductively. Alternatively, one can see that $\lambda \in T(x)$ if and only if $\frac{1}{\lambda} \in T(\frac{1}{x})$ by noting that the affine map $(x, y) \rightarrow (\frac{1}{x}, \frac{y}{x^2\sqrt{\lambda}})$ extends to an isomorphism between the elliptic curves E_λ and $E_{\frac{1}{\lambda}}$. \square

The following theorem is a restatement of [Sto14, Theorem 3]. Here we provide a different, shorter proof.

Theorem 4.2. *Let $\alpha \in \mathbb{C}_2 \setminus \{0, 1\}$ and $\lambda \in T(\alpha) \setminus \{\alpha\}$. If $|\alpha| \leq 1$, then $|\alpha^2 - \lambda| < 1$. If $|\alpha| > 1$, then $|\lambda| > 1$.*

Proof. Consider $\alpha \in \mathbb{C}_2$ with $|\alpha| \leq 1$ and let $\lambda \in T(\alpha) \setminus \{\alpha\}$. First we will show that $|\lambda| \leq 1$. Assume, so as to derive a contradiction, that $|\lambda| > 1$. Then, we have

$$|f_\lambda(\alpha)| = \frac{4|\lambda|^2}{|\alpha||\alpha - 1||\lambda|} \geq 4|\lambda|.$$

Inductively, we get $|f_\lambda^{n+1}(\alpha)| = 4|f_\lambda^n(\alpha)|$ for all $n \geq 1$. Since $\lambda \in T(\alpha)$, this implies $f_\lambda(\alpha) = \infty$ which in turn contradicts the fact that $\lambda \neq \alpha$. Therefore, we must have $|\lambda| \leq 1$. To prove $|\alpha^2 - \lambda| < 1$, let us assume the opposite and see what happens. Using the fact $|\lambda| \leq 1$, we have

$$|f_\lambda(\alpha)| = \frac{4|\alpha^2 - \lambda|^2}{|\alpha||\alpha - 1||\alpha - \lambda|} \geq 4.$$

Again, inductively we get $|f_\lambda^{n+1}(\alpha)| = 4|f_\lambda^n(\alpha)|$ for all $n \geq 1$. Since $\lambda \in T(\alpha)$, this yields $f_\lambda(\alpha) = \infty$, contradicting the fact that $\lambda \neq \alpha$. Hence $|\alpha^2 - \lambda| < 1$.

The second part of the statement now follows by the first part and Proposition 4.1. \square

Therefore, exactly as in [Sto14, Corollary 4], denoting by ρ the natural reduction map $\mathbb{P}^1(\mathbb{C}_2) \rightarrow \mathbb{P}^1(\overline{\mathbb{F}_2})$, we get that

Corollary 4.3. *If $\alpha, \beta \in \mathbb{C}_2 \setminus \{0, 1\}$ such that $\rho(\alpha) \neq \rho(\beta)$, then $T(\alpha) \cap T(\beta) \subset \{\alpha, \beta\}$.*

For examples of α, β such that $T(\alpha) \cap T(\beta)$ is empty or has exactly one or two elements, we refer the reader to [Sto14, Example 5].

Now we aim to strengthen this result. In particular we provide an effective description of $T(\alpha) \cap T(\beta)$ even in some cases when $\rho(\alpha) = \rho(\beta)$. To this end, we have the following.

Theorem 4.4. *If $\alpha \in \mathbb{C}_2$ satisfies $|\alpha| \leq \frac{1}{4}$, then $T(2) \cap T(\alpha) \subset \{\alpha\}$. Moreover, if $|\alpha| < \frac{1}{4}$, then $T(2) \cap T(\alpha) = \emptyset$.*

Proof. Let $\alpha \in \mathbb{C}_2$ be such that $|\alpha| \leq \frac{1}{4}$ and $\lambda \in T(2) \cap T(\alpha) \setminus \{2, \alpha\}$. In view of Theorem 4.2, we know that $|\lambda| < 1$. We claim, in fact, that the following is true.

Claim 4.5. *If $\lambda \in T(2) \setminus \{2\}$, then $|\lambda| = \frac{1}{4}$.*

Proof. Using an argument by contradiction, we will first show that $|\lambda| \geq \frac{1}{4}$. Assume that $|\lambda| < \frac{1}{4}$. Then $|f_\lambda(2)| = \frac{8|4-\lambda|^2}{|2-\lambda|} = 1$ and therefore $|(f_\lambda(2))^2 - \lambda| = 1$. However, since $\lambda \in T(f_\lambda(2)) \setminus \{f_\lambda(2)\}$, this contradicts Theorem 4.2.

A similar argument also shows that $|\lambda| \leq \frac{1}{2}$. Indeed, if $|\lambda| > \frac{1}{2}$, then we have $|f_\lambda(2)| = 8|\lambda| > 4$, which contradicts Theorem 4.2 since $\lambda \in T(f_\lambda(2)) \setminus \{f_\lambda(2)\}$ and $|\lambda| < 1$.

Now we know that $\frac{1}{4} \leq |\lambda| \leq \frac{1}{2}$. Assume that $|\lambda| > \frac{1}{4}$, so as to derive a contradiction and establish the claim. Since $|\lambda| \leq \frac{1}{2}$, we have $|f_\lambda(2)| = \frac{8|\lambda|^2}{|2-\lambda|} \geq 16|\lambda|^2 > 1$, which combined with the fact that $\lambda \in T(f_\lambda(2)) \setminus \{f_\lambda(2)\}$ contradicts Theorem 4.2. This yields the claim. \square

To finish the proof of the theorem, notice that Claim 4.5 yields that $|\lambda| = \frac{1}{4}$. This, combined with our assumption that $|\alpha| \leq \frac{1}{4}$, implies $|f_\lambda(\alpha)| = \frac{4|\lambda|^2}{|\alpha||\alpha-\lambda|} \geq 4$, which by

Theorem 4.2 contradicts the fact that $\lambda \in T(f_\lambda(\alpha)) \setminus \{f_\lambda(\alpha)\}$. Therefore, we obtain that $T(2) \cap T(\alpha) \subset \{2, \alpha\}$. We can easily see that $2 \notin T(\alpha)$, since $|\alpha| \leq \frac{1}{4}$. Thus, in fact we get that $T(2) \cap T(\alpha) \subset \{\alpha\}$. If in particular $|\alpha| < \frac{1}{4}$, then by Claim 4.5 we have $T(2) \cap T(\alpha) = \emptyset$. \square

Combining now Proposition 4.1 and Theorem 4.4, we get the following.

Corollary 4.6. *If $\beta \in \mathbb{C}_2$ satisfies $|\beta| \geq 4$, then $T(\frac{1}{2}) \cap T(\beta) \subset \{\beta\}$. Moreover, if $|\beta| > 4$, then $T(\frac{1}{2}) \cap T(\beta) = \emptyset$.*

5. OTHER FAMILIES

We will now consider

$$f_\lambda(z) = \frac{z^d + \lambda}{pz},$$

where $d \geq 2$ and $p \in \mathbb{Z}$ prime. Our method will give results of flavor similar to the results in Section 4 for this family of rational maps. However, we find it worthwhile to mention that the above family is not a Lattès family. To see this note that for all $\lambda \in \mathbb{C}$ the maps f_λ have an attracting fixed point in the topology induced by the standard complex absolute value; when $d > 2$ we have that ∞ is a fixed critical point and when $d = 2$ the points $\pm \sqrt{\frac{\lambda}{p-1}}$ are fixed points with multiplier $\frac{2-p}{p}$. On the other hand, for Lattès maps all periodic points are repelling and dense in $\mathbb{P}^1(\mathbb{C})$, as illustrated by the fact that the Julia set of a Lattès map is the entire Riemann sphere [S07, Theorem 1.43]. For a definition of the Julia set of a rational map we refer the reader to [S07].

In this section $|\cdot|_p$ will denote the p -adic absolute value on \mathbb{C}_p , with $|p|_p = \frac{1}{p}$. We write $T(\alpha) = \{\lambda \in \mathbb{C}_p : \alpha \text{ is preperiodic for } f_\lambda\}$. We note that 0 is a persistently preperiodic point for the family of rational maps f_λ where $\lambda \in \mathbb{C}_p$. Therefore, in the following we consider $\alpha \neq 0$.

Theorem 5.1. *Let $\alpha \in \mathbb{C}_p \setminus \{0\}$ with $|\alpha|_p \leq 1$ and let $\lambda \in T(\alpha)$. Then $|\alpha^d + \lambda|_p < 1$. If on the other hand we have $|\alpha|_p > 1$, then $|\lambda|_p > 1$.*

Proof. Consider $\alpha \in \mathbb{C}_p \setminus \{0\}$ with $|\alpha|_p \leq 1$ and let $\lambda \in T(\alpha)$. First, we will show that $|\lambda|_p \leq 1$. Assume to the contrary that $|\lambda|_p > 1$. Then, we have

$$|f_\lambda(\alpha)|_p = \frac{p|\lambda|_p}{|\alpha|_p} \geq p|\lambda|_p > p.$$

Inductively, we get that $|f_\lambda^{n+1}(\alpha)|_p = p|f_\lambda^n(\alpha)|_p^{d-1}$ for all $n \geq 1$. Since $f_\lambda(\alpha) \neq \infty$ and $d \geq 2$, this contradicts the fact that $\lambda \in T(\alpha)$. Therefore we must have $|\lambda|_p \leq 1$. Next we prove that $|\alpha^d + \lambda|_p < 1$.

Assume, for the sake of contradiction, that $\lambda \in T(\alpha)$ and $|\alpha^d + \lambda|_p \geq 1$. Then we have $|f_\lambda(\alpha)|_p \geq p$. Inductively this implies $|f_\lambda^{n+1}(\alpha)|_p = p|f_\lambda^n(\alpha)|_p^{d-1}$ for all $n \geq 1$. Since $\lambda \in T(\alpha)$ and $d \geq 2$, this yields $f_\lambda(\alpha) = \infty$, contradicting the fact that $\alpha \neq 0$. Hence $|\alpha^d + \lambda|_p < 1$.

We will now prove the second part of the statement. Assume that $|\alpha|_p > 1$ and that $\lambda \in T(\alpha)$. We will see that $|\lambda|_p > 1$. Assume, to the contrary, that $|\lambda|_p \leq 1$. Then, $|f_\lambda(\alpha)|_p = p|\alpha|_p^{d-1}$ and inductively $|f_\lambda^{n+1}(\alpha)|_p = p|f_\lambda^n(\alpha)|_p^{d-1}$ for all $n \in \mathbb{N}$. This, combined with $|\alpha|_p > 1$, contradicts our assumption that $\lambda \in T(\alpha)$. Therefore, $|\lambda|_p > 1$. The proof follows. \square

Now, if we denote by ρ the natural reduction map $\mathbb{P}^1(\mathbb{C}_p) \rightarrow \mathbb{P}^1(\overline{\mathbb{F}_p})$, we get the following theorem.

Theorem 5.2. *If $\alpha, \beta \in \mathbb{C}_p \setminus \{0\}$ are such that $\rho(\alpha^d) \neq \rho(\beta^d)$, then $T(\alpha) \cap T(\beta) = \emptyset$.*

To highlight the difference between the results in this section and in Section 2, we find it worthwhile to point out the following.

Remark 5.3. Recall that in Section 2, the map $f(z) = \frac{z^4 - 8tz}{4(z^3 + t)} \in \mathbb{C}_2(t)(z)$ has the following properties: It is isotrivial, and it is conjugate to the map $g(z) = \frac{z^4 - 2z}{4z^3 + 1}$ with 2-adic good reduction. For the map $f(z) = \frac{z^d + t}{pz} \in \mathbb{C}_p(t)(z)$ in this section, these properties are true only when $(d, p) = (2, 2)$, in which case for the choice $L(z) = 2z - 1$, we have

$$L^{-1} \circ g \circ L(z) = \frac{z^2}{2z - 1}, \text{ where } g(z) = \frac{z^2 + 1}{2z}$$

and the map $\frac{z^2}{2z-1}$ has 2-adic good reduction. If $d = 2$ but $p \neq 2$, then the map $f(z) = \frac{z^2 + t}{pz} \in \mathbb{C}_p(t)(z)$ is still isotrivial. Indeed, for $M(z) = t^{1/2}z$, we have $M^{-1} \circ f \circ M(z) = \frac{z^2 + 1}{pz} \in \mathbb{C}_p(z)$. However, $g(z) = \frac{z^2 + 1}{pz} \in \mathbb{C}_p(z)$ is not $\text{PGL}(2, \mathbb{C}_p)$ -conjugate to a map with good reduction, since it has two p -adically repelling fixed points $\pm \frac{1}{\sqrt{p-1}}$ with multiplier $\frac{2-p}{p}$, [RB14, Theorem]. Finally, if $d \geq 3$, then the map $f(z) = \frac{z^d + t}{pz} \in \mathbb{C}_p(t)(z)$ is not isotrivial.

ACKNOWLEDGEMENTS

I would like to thank Dragos Ghioca for suggesting the question and helping with exposition, and for insightful comments and enlightening conversations. I would also like to thank Rob Benedetto and Khoa Nguyen for insightful comments. Finally, I would like to thank the anonymous referee for various comments and suggestions that greatly improved this article.

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